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The quantum deformation of SU(1, 1) as the dynamical symmetry of the anharmonic oscillator

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Abstract. As an application of q-deformed algebras to standard quantum mechanics, we show that the SU(1,1) dynamical symmetry of the quantum harmonic oscillator is deformed, in the first order of approximation, to the dynamical symmetry defined by the quantized universal enveloping algebra of SU(1,1) when the harmonic potential is perturbed with a potential in x^4. The resolution of the anharmonic oscillator is carried out algebraically in terms of generalized lowering and rising operators.

1. Introduction

Recently there has been a growing interest in the study and characterization of quantum groups and quantized universal enveloping algebras (or QUE algebras), due to their applications in contexts like integrable systems, the inverse scattering problem and conformal field theory (see [1–3] and references therein). To some extent, these applications of such QUE algebras in physics currently seem very technical, and the purpose of this paper is to show that, in fact, such structures may appear, playing an underlying role, in more familiar contexts like standard quantum mechanics.

To this end, after reviewing the discrete series representations of the QUE algebra of SU(1, 1), we show that the dynamical symmetry SU(1, 1) of the standard quantum harmonic oscillator (QHO)‡ is deformed, to the first order of approximation, to the dynamical symmetry described by the QUE algebra of SU(1, 1) when the harmonic potential is subject to a suitable perturbation. In turn, such a deformation of the dynamical symmetry will provide us with a natural algebraic framework in which the resolution of the QHO perturbed with a potential of the form $V(x) \propto x^4 + K$ can be carried out algebraically.

2. The QUE algebra of SU(1, 1)

Let us consider the SU(1, 1) version of the QUE algebra introduced independently by

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‡ To avoid misunderstanding, let us stress here that we will consider the standard QHO, and not the q-deformed analogue introduced by Biedenharn and Macfarlane in [4].

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Woronowicz and Witten [2]:

$$
\begin{aligned}
&q\mathcal{H}A_+ - q^{-1}A_+\mathcal{H} = 2A_+ \\
&q^{-1}\mathcal{H}A_- - qA_-\mathcal{H} = -2A_- \\
&q^{-2}A_+A_- - q^2A_-A_+ = -4\mathcal{H}.
\end{aligned}
\tag{2.1}
$$

Here $\mathcal{H}^\dagger = \mathcal{H}$, $A_\pm^\dagger = A_\mp$, and q is the deformation parameter, which we take to be real. We have rescaled the generators for reasons that will become clear later. The algebra (2.1) coincides with the SU(1,1) invariant subalgebra of a q-deformation of the Virasoro algebra proposed in [5] and [6] and considered in [7], and it can be expressed in terms of true commutators with respect to a non-associative graded product (see [7]).

Some comments about the QUE algebra (2.1) are in order. First, note that the QUE algebra of SU(1,1) is isomorphic to the QUE algebra of SU(2). Indeed, if we define the algebra $\mathcal{G}_q \equiv \mathrm{SU}_q(1,1) = SU_{-q}(2)$, then

$$
\lim_{q\to 1}\mathcal{G}_q = \mathrm{SU}(1,1) \qquad \lim_{q\to -1}\mathcal{G}_q = \mathrm{SU}(2).
\tag{2.2}
$$

Second, a more general deformed algebra is obtained by replacing the third relationship in (2.1) by

$$
q^{-2a}A_+A_- - q^{2a}A_-A_+ = -4\mathcal{H}.
\tag{2.3}
$$

For $a = 1$, (2.3) reproduces (2.1), i.e. the Witten deformation in the strict sense, while for $a = -1$ we obtain the Woronowicz deformation [2]. Note, however, that replacing the third relationship in (2.1) by (2.3) is merely a redefinition of the operators $A_\pm$. For simplicity, we consider here the case $a = 1$. In the last section we shall give the modifications to our results for general values of the parameter a.

Given a realization of the algebra (2.1), and putting $q = \mathrm{e}^{\epsilon\beta}$, where β is a 'coupling constant', we can assume, for $\epsilon \ll 1$, the following decomposition of the generators $\{\mathcal{H}, A_\pm\}$†:

$$
\begin{aligned}
&A_\pm = A_\pm^0 + \epsilon A'_\pm + \mathcal{O}(\epsilon^2) \\
&\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{H}' + \mathcal{O}(\epsilon^2).
\end{aligned}
\tag{2.4}
$$

The zero-order operators $\{\mathcal{H}_0, A_\pm^0\}$ generate the 'zero-order' algebra

$$
\begin{aligned}
&[\mathcal{H}_0, A_\pm^0] = \pm 2A_\pm^0 \\
&[A_+^0, A_-^0] = -4\mathcal{H}_0
\end{aligned}
\tag{2.5}
$$

i.e. the algebra of SU(1,1). This is a consequence of the so called 'correspondence principle' for quantized algebras, which states that the quantized or q-deformed version $\mathcal{G}_q$ of an algebra $\mathcal{G}_0$ has to contain the algebra $\mathcal{G}_0$ in the limit $q \to 1$ (or $\epsilon \to 0$). Now, the first-order family of operators $\{\mathcal{H}', A'_\pm\}$ defines the 'first-order' algebra associated

† We consider here a realization of (2.1) for which this decomposition is well defined.

with the QUE algebra of SU(1,1) and obtained from (2.1) by retaining only the terms linear in ϵ:

$$\begin{aligned}
&\beta\{\mathcal{H}_0, A_+^0\} + [\mathcal{H}', A_+^0] + [\mathcal{H}_0, A_+'] = 2A_+' \\
&\beta\{\mathcal{H}_0, A_-^0\} + [A_-^0, \mathcal{H}'] + [A_-', \mathcal{H}_0] = 2A_-' \\
&-2\beta\{A_-^0, A_+^0\} + [A_+', A_-^0] + [A_+^0, A_-'] = -4\mathcal{H}'.
\end{aligned} \tag{2.6}$$

Note that this generalized commutation algebra (2.6) is coupled to the algebra of SU(1,1), (2.5).

The discrete representations of the algebra SU(1,1), equations (2.5)—or more precisely, of its universal covering group [8]—are determined by a single number k; for fixed k, the orthonormal basis vectors of the representation, $\{|k;n\rangle_0\}$, are labelled by an integer $n \geq 0$, and they satisfy the relationships:

$$\begin{aligned}
&\mathcal{H}_0|k;n\rangle_0 = 2(k+n)|k;n\rangle_0 \qquad C_2^0|k;n\rangle_0 = k(k-1)|k;n\rangle_0 \\
&A_-^0|k;0\rangle_0 = 0 \\
&A_+^0|k;n\rangle_0 = c_n^0|k;n+1\rangle_0 \qquad A_-^0|k;n\rangle_0 = c_{n-1}^0|k;n-1\rangle_0
\end{aligned} \tag{2.7}$$

where $C_2^0 = \frac{1}{4}\mathcal{H}_0^2 - \frac{1}{8}(A_+^0A_-^0 + A_-^0A_+^0)$ is the Casimir operator of SU(1,1), and

$$c_n^0 = 2\sqrt{(n+1)(2k+n)}. \tag{2.8}$$

On the other hand, assuming the existence of a highest-weight vector $|k_q;0\rangle$ and following the standard method, it is easily seen that, as an extension of the results in [5, 7, 9], the quantized algebra (2.1) admits discrete series representations. The representations are determined by a quantity k_q, and the vectors are again labelled by an integer $n \geq 0$, $\{|k_q;n\rangle\}$; these orthonormal vectors satisfy the properties

$$\begin{aligned}
&\mathcal{H}|k_q;n\rangle = h(n)|k_q;n\rangle \\
&A_-|k_q;0\rangle = 0 \\
&A_+|k_q;n\rangle = c_n|k_q;n+1\rangle \qquad A_-|k_q;n\rangle = c_{n-1}|k_q;n-1\rangle
\end{aligned} \tag{2.9}$$

where

$$\begin{aligned}
&h(n) = 2\left(k_q - \frac{1}{q-q^{-1}}\right)q^{-2n} + \frac{2}{q-q^{-1}} \\
&(c_n)^2 = \frac{8q}{(1-q^2)(1-q^4)}(q^2 - q^{-2n})\Big(1 - \big(1 - k_q(q-q^{-3})\big)q^{-2n}\Big).
\end{aligned} \tag{2.10}$$

The expressions for $h(n)$ and c_n reduce, in the limit $q \to 1$, to their q-classical values, i.e. we recover the $\mathcal{H}_0$ eigenvalues in (2.7) and the expression for c_n^0 in (2.8).

Some words about the Casimir operator are in order. There are two inequivalent ways of defining the Casimir operator $\mathcal{C}_2$ in the deformed algebra. Either we take $\mathcal{C}_2$ to satisfy the deformed commutators:

$$\begin{aligned}
&q^{\pm 2}\tilde{\mathcal{C}}_2 A_\pm = q^{\mp 2} A_\pm \tilde{\mathcal{C}}_2 \\
&\tilde{\mathcal{C}}_2\mathcal{H} = \mathcal{H}\tilde{\mathcal{C}}_2
\end{aligned} \tag{2.11}$$

or we take $\mathcal{C}_2$ to commute with the generators $\{A_\pm, \mathcal{H}\}$ in the strict sense:

$$[\mathcal{C}_2, A_\pm] = [\mathcal{C}_2, \mathcal{H}] = 0. \tag{2.12}$$

In the first case there are two operators satisfying (2.11):

$$\tilde{\mathcal{C}}_2 = \tfrac{1}{4}\mathcal{H}^2 - \tfrac{1}{8}(qA_-A_+ + q^{-1}A_+A_-) \qquad \tilde{\mathcal{C}}_2' = \left(\frac{q-q^{-1}}{2}\mathcal{H} - 1\right)^2 \tag{2.13}$$

while in the second case $\mathcal{C}_2$ is given by

$$\mathcal{C}_2 = \frac{\tilde{\mathcal{C}}_2}{\tilde{\mathcal{C}}_2'}. \tag{2.14}$$

Note that in a given representation the Schurr lemma can only be applied to $\mathcal{C}_2$, and indeed:

$$\mathcal{C}_2|k_q; n\rangle = \frac{k_q(k_q - q^{-1})}{\left[(q-q^{-1})k_q - 1\right]^2}|k_q; n\rangle. \tag{2.15}$$

The eigenvalues of $|k_q; n\rangle$ with respect to the operators $\tilde{\mathcal{C}}_2$ and $\tilde{\mathcal{C}}_2'$, however, are functions of both k_q and n.

Now invoking the correspondence principle for QUE algebras, each discrete series representation of SU(1, 1) is expected to be the $q \to 1$ limit of some discrete representation of (2.1) of the form (2.9). Assuming $k_q = k + \epsilon\beta k' + \mathcal{O}(\epsilon^2)$, we get to the first order of approximation

$$\begin{aligned} h(n) &= h_0(n) + \epsilon h'(n) + \mathcal{O}(\epsilon^2) \\ c_n &= c_n^0 + \epsilon c_n' + \mathcal{O}(\epsilon^2) \end{aligned} \tag{2.16}$$

where

$$\begin{aligned} h_0(n) &= 2(k+n) \qquad h'(n) = 2\beta\big(k' - n(2k+n)\big) \\ c_n' &= -2\beta(c_n^0)^{-1}(n+1)\big(2n^2 + 6kn + 4k + n - 2k'\big). \end{aligned} \tag{2.17}$$

The states $|k_q; n\rangle$ can be decomposed as

$$|k_q; n\rangle = |k; n\rangle_0 + \epsilon|k, k'; n\rangle' + \mathcal{O}(\epsilon^2). \tag{2.18}$$

Now taking (2.9) into account, it follows that the vectors $\{|k; n\rangle_0\}$ verify the relationships in (2.7), and for the vectors $\{|k, k'; n\rangle'\}$ we get:

$$\begin{aligned} &\mathcal{H}_0|k, k'; n\rangle' + \mathcal{H}'|k; n\rangle_0 = h_0(n)|k, k'; n\rangle' + h'(n)|k; n\rangle_0 \\ &A_-^0|k, k'; 0\rangle' + A_-'|k; 0\rangle_0 = 0 \\ &A_+^0|k, k'; n\rangle' + A_+'|k; n\rangle_0 = c_n^0|k, k'; n+1\rangle' + c_n'|k; n+1\rangle_0 \\ &A_-^0|k, k'; n\rangle' + A_-'|k; n\rangle_0 = c_{n-1}^0|k, k'; n-1\rangle' + c_{n-1}'|k; n-1\rangle_0. \end{aligned} \tag{2.19}$$

In terms of the highest-weight vector $|k;0\rangle_0 + \epsilon|k,k';0\rangle'$, the vectors of the representation, $\{|k;n\rangle_0 + \epsilon|k,k';n\rangle'\}$, are completely determined by the relationships

$$|k;n\rangle_0 = \left(\prod_{i=0}^{n-1} c_i^0\right)^{-1} (A_+^0)^n |k;0\rangle_0 \tag{2.20}$$

$$|k,k';n\rangle' = \left(\prod_{i=0}^{n-1} c_i^0\right)^{-1} \left[\sum_{j=0}^{n-1}\left((A_+^0)^j A'_+ (A_+^0)^{n-j-1} - \frac{c'_j}{c_j^0}(A_+^0)^n\right) \times |k;0\rangle_0 + (A_+^0)^n |k,k';0\rangle'\right]. \tag{2.21}$$

While the discrete series representations of SU$(1,1)$ are labelled by a single quantity k, the representations of the same algebra coupled to the algebra (2.6) are labelled by k and an additional quantity k'. Moreover, since we have adopted an orthonormal basis for every value of q, we have $\langle k_q;n|k_q;n\rangle = 1 = {}_0\langle k;n|k;n\rangle_0$; assuming that the vectorial space in which the representation acts is real, it follows that

$${}_0\langle k;n|k,k';n\rangle' = 0. \tag{2.22}$$

In the following, after a brief review of the dynamical symmetry of the QHO, we are going to give a realization of the commutator algebras (2.5) and (2.6). Such a realization will determine discrete series representations of the first-order QUE algebra of SU$(1,1)$, (2.7) and (2.19), as the dynamical symmetry of the anharmonic oscillator. Hence, the energy levels of the anharmonic oscillator will be given, to the first order, by the eigenvalues of $\mathcal{H}$, equation (2.17), and the corresponding eigenvectors will be completely determined by (2.20) and (2.21).

3. The harmonic oscillator and its dynamical symmetry

In natural units $m = \omega = \hbar = 1$, the one-dimensional harmonic oscillator is described by the Hamiltonian $\mathcal{H}_0 = a_+a_- + \frac{1}{2}$, where $a_\pm$ stand for the bosonic creation and annihilation operators satisfying the commutation relation $[a_-, a_+] = 1$. Explicitly

$$\mathcal{H}_0 = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2 \qquad a_\pm = \frac{1}{\sqrt{2}}\big(x \mp \partial_x\big). \tag{3.1}$$

The orthonormal $\mathcal{H}_0$-eigenstates $\phi_n(x)$, with $\mathcal{H}_0$-eigenvalues $(n+\frac{1}{2})$, are

$$\phi_n(x) = N_n H_n(x) \mathrm{e}^{-x^2/2} \tag{3.2}$$

where $H_n = (-1)^n \mathrm{e}^{x^2}\partial_x^n \mathrm{e}^{-x^2}$ denotes the nth Hermite polynomial, and $N_n = \big(\sqrt{\pi}2^n n!\big)^{-1/2}$ is a renormalization constant. Since the states $\phi_n(x)$ have parity n, $\phi_n(-x) = (-1)^n\phi_n(x)$, the vector basis decomposes into an even and odd sector.

The Hamiltonian $\mathcal{H}_0$ and the bilinear operators $A_\pm^0 \equiv a_\pm^2$

$$A_\pm^0 = \tfrac{1}{2}\big(\partial^2 \mp 2x\partial + (x^2 \mp 1)\big) \tag{3.3}$$

satisfy the commutation relations (2.5), and they generate the algebra of SU(1, 1). It is straightforward to check that in fact the states $\phi_n(x)$ for even and odd values of n transform by an irreducible representation in discrete series of SU(1, 1), with $k = \frac{1}{4}$ and $k = \frac{3}{4}$ respectively, and

$$|\tfrac{1}{4}; n\rangle_0 = \phi_{2n}(x) \qquad |\tfrac{3}{4}; n\rangle_0 = \phi_{2n+1}(x). \tag{3.4}$$

Hence, these states verify (2.7) and (2.20), and SU(1, 1) constitutes the dynamical symmetry of the QHO.

4. The anharmonic oscillator

From the considerations of the former sections, we can consider the conditions under which the SU(1, 1) dynamical symmetry of the QHO can be extended—or more precisely, deformed—to the symmetry defined by the generalized commutation relations (2.6), together with (2.5). To achieve this, we 'deform' $\mathcal{H}_0$ and $A^0_\pm$ given in (3.1) and (3.3) to the operators $\mathcal{H}$ and $A_\pm$ given in (2.4). We choose adequate ansätze for the operators $A'_\pm$, with $(A'_\pm)^\dagger = A'_\mp$, and—since we want to maintain the Hamiltonian character of $\mathcal{H}_0$—we require $\mathcal{H}'$ to be a function of the variable x. We have found the following solution:

$$A'_\pm = \frac{\beta}{12}(\mp 6x\partial^3 + (12x^2 \mp 9)\partial^2 + (\mp 2x^3 + 24x)\partial + (-4x^4 \mp 3x^2 + 6)) \tag{4.1}$$

provided $\mathcal{H}'$ is identified with the potential†:

$$\mathcal{H}' = -\frac{\beta}{3}\left(x^4 - \frac{3}{2}\right). \tag{4.2}$$

The resulting Hamiltonian is now

$$\mathcal{H} = \mathcal{H}_0 + \epsilon\mathcal{H}' = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2 - \epsilon\frac{\beta}{3}\left(x^4 - \frac{3}{2}\right) + \mathcal{O}(\epsilon^2) \tag{4.3}$$

where β plays the role of coupling constant.

Once we have obtained a realization of the generalized commutator algebra (2.5) and (2.6), let us consider the space of states of the representation for such an operator algebra. Since the first-order operators $A'_\pm$ and $\mathcal{H}'$ preserve the parity of the zero-order representations, let us consider both sectors separately.

Even sector, $k = \frac{1}{4}$. From (2.19) we get the two following conditions for the highest-weight state of the representation (the ground state of $\mathcal{H}$):

$$A^0_-|0\rangle' + A'_-|0\rangle_0 = 0 \qquad \mathcal{H}_0|0\rangle' + \mathcal{H}'|0\rangle_0 = 2k|0\rangle' + 2\beta k'|0\rangle_0. \tag{4.4}$$

† In fact, the general solution for $\mathcal{H}'$ also contains a term of the form gx^{-2} that defines the singular oscillator; in this case, much of the results presented here can be extended to the *odd* sector of the QHO, following the considerations of [8].

Replacing $|0\rangle_0$ by the wavefunction $\phi_0(x)$, and the operators by their explicit expressions, (4.4) fixes $k' = \frac{1}{8}$ and determines the explicit form for the highest-weight state $|0\rangle = |0\rangle_0 + \epsilon|0\rangle' + \mathcal{O}(\epsilon^2)$:

$$|0\rangle_0 = \phi_0(x) \qquad |0\rangle' = \frac{\beta}{12}\left(x^4 + 3x^2 - \frac{9}{4}\right)\phi_0(x). \tag{4.5}$$

The energy of this state is $h(0) = 2(k + \epsilon\beta k') + \mathcal{O}(\epsilon^2) = \frac{1}{2} + \epsilon\beta/4 + \mathcal{O}(\epsilon^2)$. In solving (4.4), we have imposed the supplementary condition ${}_0\langle 0|0\rangle' = 0$.

Now, the energy of the states $|n\rangle = |n\rangle_0 + \epsilon|n\rangle'$, $E_n = h(n)$, can be read directly from (2.17):

$$E_n = \left(\frac{1}{2} + 2n\right) - \epsilon\frac{\beta}{4}(8n^2 + 4n - 1) + \mathcal{O}(\epsilon^2) \tag{4.6}$$

and (2.20) and (2.21) determine the explicit form of these states

$$|n\rangle_0 = \left(\prod_{i=0}^{n-1} c_i^0\right)^{-1} (A_+^0)^n \phi_0(x) = \phi_{2n}(x) \tag{4.7}$$

$$|n\rangle' = \left(\prod_{i=0}^{n-1} c_i^0\right)^{-1} \left[\sum_{j=0}^{n-1}\left((A_+^0)^j A_+' (A_+^0)^{n-j-1} - \frac{c_j'}{c_j^0}(A_+^0)^n\right)\right.$$
$$\left. \times\, \phi_0(x) + (A_+^0)^n \frac{\beta}{12}\left(x^4 + 3x^2 - \frac{9}{4}\right)\phi_0(x)\right].$$

Making use of the properties of the Hermite polynomials [10], we have obtained the following simple general expression

$$|n\rangle = \left(1 + \epsilon\beta\frac{F_{2n}(x)}{H_{2n}(x)}\right)\phi_{2n}(x) + \mathcal{O}(\epsilon^2) \tag{4.8}$$

where

$$F_p(x) = \frac{1}{12}\left[\left(x^4 + 3(p+1)x^2 - \frac{9}{4}(2p+1)\right)H_p(x) - p\left(2x^3 + 3(2p+1)x\right)H_{p-1}(x)\right]. \tag{4.9}$$

As a check, it is straightforward to see that $|n\rangle'$ verifies the first of the relationships in (2.19). The fact that in solving (4.4) we have imposed the condition ${}_0\langle 0|0\rangle' = 0$ guarantees that the relationships (2.22) are satisfied for every n, i.e. by construction ${}_0\langle n|n\rangle' = 0$.

Odd sector, $k = \frac{3}{4}$. In this sector $|0\rangle_0 = \phi_1(x)$. Proceeding in the same way, (4.4) fixes $k' = -\frac{3}{8}$, and yield for the highest-weight state of the representation

$$|0\rangle = \left(1 + \epsilon\frac{\beta}{12}\left(x^4 + 5x^2 - \frac{45}{4}\right)\right)\phi_1(x) + \mathcal{O}(\epsilon^2) \tag{4.10}$$

with energy $h(0) = 2(k + \beta k') = 3/2 - \epsilon 3\beta/4 + \mathcal{O}(\epsilon^2)$. For the states $|n\rangle$ we get the expression

$$|n\rangle = \left(1 + \epsilon\beta \frac{F_{2n+1}(x)}{H_{2n+1}(x)}\right) \phi_{2n+1}(x) + \mathcal{O}(\epsilon^2) \tag{4.11}$$

with energies $E_n = h(n)$

$$E_n = \left(\frac{3}{2} + 2n\right) - \epsilon\frac{\beta}{4}(8n^2 + 12n + 3) + \mathcal{O}(\epsilon^2). \tag{4.12}$$

It is straightforward to check that these results coincide with the ones we would expect by applying the standard perturbation method to the perturbed Hamiltonian. In particular, the first-order corrections to the energy levels of the QHO due to a potential of the form $V(x) = -\beta/3(x^4 - \frac{3}{2})$ are given by

$$\langle\phi_m|V(x)|\phi_m\rangle = -\beta/12\langle\phi_m|(a_- + a_+)^4 - 6|\phi_m\rangle = -\beta/4(2m^2 + 2m - 1). \tag{4.13}$$

Now putting $m = 2n$ and $m = 2n + 1$ we recover (4.6) and (4.12), respectively. It can be also seen that the expressions (4.8) and (4.11) reproduce the corresponding ones obtained by standard perturbation techniques:

$$|n\rangle_0 \to |n\rangle_0 + \epsilon\beta \sum_{j\neq n} \frac{{}_0\langle j|V|n\rangle_0}{E_n - E_j}|j\rangle_0. \tag{4.14}$$

5. Conclusions and comments

As an illustration of the fact that the representation theory of QUE algebras is in correspondence to that of their 'q-classical' ($q \to 1$ limit) algebras, we have shown explicitly that the dynamical symmetry of the one-dimensional QHO is the q-classical limit of the (first-order) QUE SU(1,1) dynamical symmetry of the anharmonic oscillator. Roughly speaking, the first-order q-deformation of SU(1,1) to its QUE version entails the deformation of the Hilbert space of the QHO to that of the anharmonic oscillator.

By exploiting this underlying symmetry of the anharmonic oscillator, we have endowed the perturbative calculation of the energies and—especially—of the wavefunctions of the anharmonic oscillator with an elegant and simple algebraic structure. In particular, under such a perturbation in the harmonic potential, the even (odd) sector of the QHO, that provides a representation of SU(1,1) with $k = \frac{1}{4}$ ($k = \frac{3}{4}$), displays a representation of the first-order QUE algebra of SU(1,1) corresponding to the values $k = \frac{1}{4}, k' = \frac{1}{8}$ ($k = \frac{3}{4}, k' = -\frac{3}{8}$).

The results presented here in detail can be extended to more general circumstances. Under the transformation $A_\pm \to q^{(1-a)}A_\pm$, for instance, the third relationship in (2.1) becomes (2.3). In such a case we have $A^0_\pm \to A^0_\pm$ and $A'_\pm \to \beta(1-a)A^0_\pm + A'_\pm$, while the potential becomes

$$\mathcal{H}' = -\frac{\beta}{3}\left(x^4 - \frac{3}{2}\right) \to -\frac{\beta}{3}\left(x^4 - \frac{3a}{2}\right). \tag{5.1}$$

Henceforth, the method also applies to a general perturbation of the form $(x^4 + K)$ for an arbitrary constant K.

The procedure presented here can also be further extended to higher orders. To second order, for instance, the harmonic Hamiltonian is deformed to the following one

$$\mathcal{H} = -\frac{1}{2}\partial^2 + \frac{1}{2}x^2 - \epsilon\frac{\beta}{3}\left(x^4 - \frac{3}{2}\right) - \epsilon^2\frac{\beta^2}{18}\left(\frac{63}{4}x^2 - \frac{23}{5}x^6\right) + \mathcal{O}(\epsilon^3) \tag{5.2}$$

and it can be 'solved' to second order following the algebraic method considered here.

We may expect that some particular realization of the QUE algebra of SU(1, 1) should constitute the *exact* dynamical symmetry of an ordinary quantum system described by a Hamiltonian of the form $\mathcal{H} = -\frac{1}{2}\partial^2 + V_q(x)$, for a suitable one-dimensional potential $V_q(x)$. The corresponding energy levels would be given by $h(n)$ in (2.10). Note that $h(n) \simeq q^{-2n}$ and therefore q cannot be smaller than one, as otherwise the energy spectrum would be increasing exponentially†. Therefore, for $V_q(x)$ to be a realistic physical potential, the deformation parameter has to be subject to the condition $q \geq 1$. The asymptotic behaviour in the domain $0 \leq (q-1) \ll 1$ of such a deformed potential would be

$$V_q(x) = \frac{1}{2}x^2 - \frac{\ln q}{3}(x^4 - \frac{3}{2}) - \frac{(\ln q)^2}{18}\left(\frac{63}{4}x^2 - \frac{23}{5}x^6\right) + \dots \tag{5.3}$$

It would be interesting to get the specific form of such a class of one-dimensional potentials (if any) and to investigate their possible physical relevance. These questions remain to be investigated.

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† This is not possible, as in quantum mechanics no realistic one-dimensional potential exists whose energy levels grow faster than n^2, which is the limit case of the infinitely deep square well potential. I thank Ph Spindel for pointing this out.